Problems on Continuous Random variable

2. If the joint pdf of (x,y) is given by $f(x,y) = \begin{cases} x^2+\frac{xy}{3}, & 0<x<1,\ 0<y<2 \\ 0 & \text{otherwise} \end{cases}$

Find $P(x>\frac{1}{2})$, $P(y<\frac{1}{2} \mid x<\frac{1}{2})$ and $P(y<x)$.

Solution:

To find $P(x>\frac{1}{2})$

Let $P(x>\frac{1}{2}) = \int_{\frac{1}{2}}^{\infty} f(x)\,dx.$

Hence $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{0}^{2} \left(x^2+\frac{xy}{3}\right)dy$

$= \left(x^2y + \frac{x}{3}\frac{y^2}{2}\right)_0^2 = \left(2x^2 + \frac{x}{3}\left(\frac{4}{2}\right) - 0\right)$

$$\boxed{f(x) = 2x^2 + \frac{2x}{3}, \quad 0<x<1}$$

Now $P(x>\frac{1}{2}) = \int_{\frac{1}{2}}^{1} \left(2x^2+\frac{2x}{3}\right)dx$

$= \left(\frac{2x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}\times 3}\right)_{\frac{1}{2}}^{1}$

$= \left(\frac{2}{3}+\frac{1}{3}\right) - \left(\frac{2(\frac{1}{2})^3}{3} + \frac{(\frac{1}{2})^2}{3}\right)$

$= 1 - \left(\frac{1}{12}+\frac{1}{12}\right) = 1 - \frac{1}{6} = \frac{5}{6}$

$$\boxed{P(x>\frac{1}{2}) = 0.833}$$

(ii)

$$P(Y<1/2 \mid X<1/2) = \frac{P(Y<1/2,\ X<1/2)}{P(X<1/2)} = \frac{P(X<1/2,\ Y<1/2)}{P(X<1/2)}$$

$$= \int_0^{1/2}\int_0^{1/2} \left(x^2 + \frac{xy}{3}\right) dx\,dy$$

$$= \int_0^{1/2} \left(\frac{x^3}{3} + \frac{x^2 y}{6}\right)_0^{1/2} dy$$

$$= \int_0^{1/2} \left(\frac{1}{24} + \frac{y}{24}\right) dy$$

$$= \frac{1}{24}\left[y + \frac{y^2}{2}\right]_0^{1/2}$$

$$= \frac{1}{24}\left[\frac{1}{2} + \frac{1}{8}\right] - 0$$

$$= \frac{1}{24}\left(\frac{4+1}{8}\right) = \frac{5}{192}$$

Let $P(X<1/2) = 1 - P(X \geq 1/2) = 1 - \frac{5}{6} = \frac{1}{6}$

$$\therefore P(Y<1/2 \mid X<1/2) = \frac{\frac{5}{192}}{\frac{1}{6}} = \frac{5}{\cancel{192}_{32}} \times \frac{\cancel{6}}{1}$$

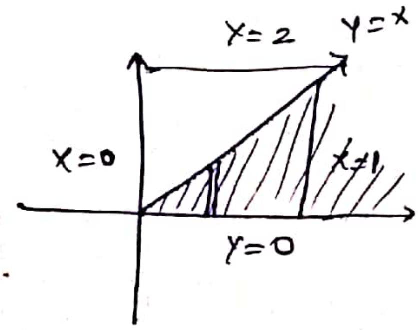
$$\boxed{P(Y<1/2 \mid X<1/2) = \frac{5}{32} = 0.156}$$

(iii)

$$P(Y<X) = \int_0^1 \int_0^x f(x,y)\,dy\,dx$$

$$= \int_0^1 \left(x^2 + \frac{xy}{3}\right) dy\,dx$$

$$= \int_0^1 \left(x^2 y + \frac{xy^2}{6}\right)_0^x dx$$



$$= \int_0^1 \left(x^3 + \frac{x^3}{6} - 0\right) dx$$

$$= \int_0^1 \frac{7x^3}{6}\, dx$$

$$= \frac{7}{6}\left(\frac{x^4}{4}\right)_0^1$$

$$\boxed{P(Y<X) = \frac{7}{24}}$$

3. If the joint pdf of (X,Y) is $f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$

Find the marginal densities of X and Y. Find also $P\left(\frac{1}{4} \le y \le \frac{3}{4}\right)$.

Soln:

Marginal density of $X = f(x) = \int_0^1 f(x,y)\,dy$

$$= \int_0^1 \frac{6}{5}(x+y^2)\,dy$$

$$= \frac{6}{5}\left(xy + \frac{y^3}{3}\right)_0^1$$

$$= \frac{6}{5}\left(x + \frac{1}{3} - 0\right)$$

$$\boxed{f(x) = \frac{6}{5}(x + 1/3)}, \quad 0 \le x \le 1$$

Marginal density of $Y = f(y) = \int_0^1 f(x,y)\,dx$

$$= \int_0^1 \frac{6}{5}(x+y^2)\,dx$$

$$= \frac{6}{5}\left(\frac{x^2}{2} + xy^2\right)_0^1$$

$$= \frac{6}{5}\left(\frac{1}{2} + y^2 - 0\right)$$

$$f(y) = \frac{6}{5}\left(\frac{1}{2} + y^2\right), \quad 0 \le y \le 1$$

(ii) $P\left(\frac{1}{4} \le y \le \frac{3}{4}\right) = \int_{1/4}^{3/4} f(y)\,dy = \int_{1/4}^{3/4} \frac{6}{5}\left(\frac{1}{2}+y^2\right)dy$

$$= \frac{6}{5}\left(\frac{1}{2}y + \frac{y^3}{3}\right)_{1/4}^{3/4}$$

$$= \frac{6}{5}\left(\frac{1}{2}\left(\frac{3}{4}-\frac{1}{4}\right) + \frac{1}{3}\left(\left(\frac{3}{4}\right)^3 - \left(\frac{1}{4}\right)^3\right)\right)$$

$$= \frac{6}{5}\left(\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{4} + \frac{1}{3}\left(\frac{\cancel{26}^{13}}{\cancel{64}_{32}}\right)\right)$$

$$= \frac{6}{5}\left(\frac{1}{4} + \frac{13}{96}\right)$$

$$= \frac{6}{5}\left(\frac{24+13}{96}\right) = \frac{\cancel{6}}{5} \times \frac{37}{\cancel{96}_{16}}$$

$$\boxed{P\left(\frac{1}{4} \le y \le \frac{3}{4}\right) = \frac{37}{80}} = 0.4625$$

3. If the joint pdf of (X,Y) is $f(x,y) = \begin{cases} e^{-(x+y)}, & x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$

find $P(X<1)$, $P(X+Y<1)$

Soln:

Let $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} e^{-(x+y)}\,dy = e^{-x}\int_0^{\infty} e^{-y}\,dy$

$$= e^{-x}\left(-e^{-y}\right)_0^{\infty}.$$

$$f(x) = e^{-x}\left(-e^{-\infty}+e^{0}\right)$$

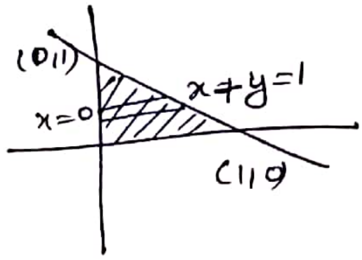
$$f(x) = e^{-x}(0+1)$$

$$\boxed{f(x) = e^{-x}, \quad x>0}$$

Let $P(X<1) = \int_0^1 f(x)\,dx = \int_0^1 e^{-x}\,dx = \left(-e^{-x}\right)_0^1$

$$= -e^{-1}+e^{0} = 1-e^{-1}$$

(ii) $$P(X+Y<1) = \int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$$



$$= \int_0^1 \int_0^{1-y} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^1 e^{-y} \left(-e^{-x}\right)_0^{1-y} dy$$

$$= \int_0^1 e^{-y} \left(-e^{-(1-y)} + e^{0}\right) dy$$

$$= \int_0^1 e^{-y} \left(-e^{-1} e^{y} + 1\right) dy$$

$$= \int_0^1 \left(-e^{-1} + e^{-y}\right) dy$$

$$= \left(-e^{-1}(y) - e^{-y}\right)_0^1$$

$$= \left(-e^{-1}(1) - e^{-1}\right) - \left(0 - e^{0}\right)$$

$$= -e^{-1} - e^{-1} + 1$$

$$= 1 - 2e^{-1}$$

4. If the joint probability distribution of (X,Y) is

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$$

(i) find Marginal densities of X & Y (ii) Are X & Y indept?

(iii) $P(1<X<3,\ 1<Y<2)$.

Soln: Let $$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

$$f(x,y) = \frac{\partial^2}{\partial x \partial y}(1-e^{-x})(1-e^{-y})$$

$$= \frac{\partial}{\partial x}(1-e^{-x}) \frac{\partial}{\partial y}(1-e^{-y})$$

$$= e^{-x} e^{-y}$$

$$\boxed{f(x,y) = e^{-x} e^{-y}, \quad x>0, y>0}$$

Let $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} e^{-x} e^{-y}\,dy$

$$= e^{-x}(-e^{-y})_0^{\infty} = e^{-x}(-e^{-\infty} + e^{0})$$

$$= e^{-x}(0+1)$$

$$\boxed{f(x) = e^{-x}, \quad x>0}$$

Let $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} e^{-x} e^{-y}\,dx = e^{-y}(-e^{-x})_0^{\infty}$

$$f(y) = e^{-y}(-e^{-\infty} + e^{0})$$

$$\boxed{f(y) = e^{-y}, \quad y>0}$$

Let $f(x) * f(y) = e^{-x} e^{-y} = f(x,y)$

$\therefore$ X & Y are independent.

(iii) $P(1<X<3, 1<Y<2) = \int_1^2 \int_1^3 e^{-x} e^{-y}\,dx\,dy$

$$= (-e^{-x})_1^3 (-e^{-y})_1^2$$

$$= (-e^{-3} + e^{-1})(-e^{-2} + e^{-1})$$

$$= (0.3181)(0.2325)$$

$$= 0.0789$$

(5) If the joint pdf of (X,Y) is $f(x,y) = \begin{cases} Kxy\,e^{-(x^2+y^2)}, & x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$

i) Find K (ii) Prove that X & Y are independent.

Soln:

To find K.

$$\text{WKT} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^{\infty}\int_0^{\infty} Kxy\,e^{-(x^2+y^2)}\,dx\,dy = 1$$

$$K\int_0^{\infty} x e^{-x^2}\,dx \int_0^{\infty} y e^{-y^2}\,dy = 1$$

Let $x^2 = u$

$2x\,dx = du$

$\Rightarrow x\,dx = \frac{du}{2}$

x	0	∞
u	0	∞

$u \to 0$ to ∞

Let $y^2 = v$

$2y\,dy = dv$

$y\,dy = \frac{dv}{2}$

y	0	∞
v	0	∞

$v \to 0$ to ∞

$$\therefore \quad K\int_0^{\infty} e^{-u}\frac{du}{2}\int_0^{\infty} e^{-v}\frac{dv}{2} = 1$$

$$\frac{K}{4}\left(-e^{-u}\right)_0^{\infty}\left(-e^{-v}\right)_0^{\infty} = 1$$

$$\frac{K}{4}\left(-e^{-\infty}+e^{0}\right)\left(-e^{-\infty}+e^{0}\right) = 1$$

$$\frac{K}{4}(0+1)(0+1) = 1$$

$$\Rightarrow \boxed{K=4}$$

$$\therefore \quad f(x,y) = \begin{cases} 4xye^{-(x^2+y^2)}, & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

Let $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} 4xye^{-(x^2+y^2)}\,dy$

$$= 4x\int_0^{\infty} ye^{-x^2}e^{-y^2}\,dy$$

$$= 4xe^{-x^2}\int_0^{\infty} ye^{-y^2}\,dy$$

$$= 4xe^{-x^2}\int_0^{\infty} e^{-v}\,\frac{dv}{2}$$

$$= 2xe^{-x^2}\left(-e^{-v}\right)_0^{\infty}$$

$$= 2xe^{-x^2}(0+1)$$

$$\boxed{\therefore f(x) = 2xe^{-x^2},\ x>0}$$

Let $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} 4xye^{-(x^2+y^2)}\,dx$

$$= \int_0^{\infty} 4xye^{-x^2}e^{-y^2}\,dx = 4ye^{-y^2}\int_0^{\infty} xe^{-x^2}\,dx$$

$$= 4ye^{-y^2}\int_0^{\infty} e^{-u}\,\frac{du}{2}$$

$$= 2ye^{-y^2}\left(-e^{-u}\right)_0^{\infty} = 2ye^{-y^2}\left(-e^{-\infty}+e^{0}\right)$$

$$\boxed{\therefore f(y) = 2ye^{-y^2},\ y>0}$$

Now $f(x)\times f(y) = 2xe^{-x^2}\times 2ye^{-y^2}, \quad x>0,\ y>0$

$$= 4xye^{-(x^2+y^2)}, \quad x>0, y>0$$

$$= f(x,y).$$

Hence X & Y are Indept.